

# EVERY FINITELY GENERATED GROUP IS WEAKLY EXACT

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**ABSTRACT.** We show that every finitely generated group admits weak analogues of an invariant expectation, whose existence characterizes exact groups. This fact has a number of applications. We show that Hopf  $G$ -modules are relatively injective, which implies that bounded cohomology groups with coefficients in all Hopf  $G$ -modules vanish in all positive degrees. We also prove a general fixed point theorem for actions of finitely generated groups on  $\ell_\infty$ -type spaces. Finally, we define the notion of weak exactness for certain Banach algebras.

In our previous work [5], we studied exact groups and their bounded cohomology. We also introduced the notion of Hopf  $G$ -modules, a class of bounded Banach  $G$ -modules which are additionally equipped with a natural representation of the algebra  $\ell_\infty(G)$ . This work initiated the consideration of  $G$ -modules with an additional representation of a  $G$ - $C^*$ -algebra as coefficients for the bounded cohomology. The techniques of [5] provided some of the key new ingredients of the characterization of amenable actions, and in particular of exact groups, in terms of bounded cohomology [1], [9].

These recent results allow one to view various amenability-like properties via bounded cohomology in a unified manner. The strength of these amenability-like properties corresponds precisely to the extent of the class of bounded  $G$ -modules for which the bounded cohomology vanishes. In Johnson's classic theorem [7] characterizing amenability, this class consists of all dual modules. Topological amenability of an action of a group  $G$  on a compact space  $X$  is detected by a subclass, the class of duals of  $\ell_1$ -geometric  $G$ -modules which are additionally equipped with a compatible representation of  $C(X)$  (see [1]). In this note we are considering the class of dual Hopf  $G$ -modules introduced in [5]. These modules correspond to  $X = \beta G$ , the Stone-Ćech compactification of  $G$ , and certain particular representations of  $C(\beta G) \simeq \ell_\infty(G)$  and constitute a subclass of the previously discussed classes of test modules.

One can similarly compare various notions of amenability using averaging operators. Amenable groups are precisely the groups for which there exists an invariant mean, a positive operator  $\ell_\infty(G) \rightarrow \mathbb{R}$  which is invariant under the group action. Exact groups are characterized by the existence of an invariant expectation [5], a map  $M : \mathcal{L}(\ell_u(G), \ell_\infty(G)) \rightarrow \ell_\infty(G)$ , where  $\mathcal{L}(X, Y)$  is the space of bounded linear operators from  $X$  to  $Y$  and  $\ell_u(G)$  is the uniform convolution algebra of  $G$ , see [5]. The invariant expectation is required to commute with the actions of  $G$ .

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The invariant expectation was the main tool used in the vanishing theorem for bounded cohomology in [5]. It also gives a convenient way to weaken or strengthen exactness by enlarging or reducing the space on which such an expectation is defined. This led us to consider a condition which we initially called *weak exactness* and which was expressed in terms of the existence of a weaker notion of an invariant expectation, defined on a space smaller than the one needed for exactness. As it turns out, this condition is rather mild.

**Theorem 1.** *A weak invariant expectation (with coefficients in any dual module) exists on every finitely generated group.*

Despite such generality weak invariant expectations turn out to be very useful. We present here three applications.

First we apply the weak invariant expectation to show that weak-\* closed Hopf  $G$ -modules are relatively injective bounded Banach  $G$ -modules (Theorem 7). In particular, this implies that bounded cohomology groups with coefficients in weak-\* closed Hopf  $G$ -modules vanish (Theorem 8) in all positive degrees. Hopf  $G$ -modules are Banach subspaces of  $\ell_\infty(G, X^*)$ , where  $X$  is a bounded  $G$ -module, which are closed with respect to both the natural action of  $G$  and the multiplicative action of  $\ell_\infty(G)$  and are additionally closed in the weak-\* topology. In [5] we conjectured that vanishing of bounded cohomology with coefficients in weak-\* closed Hopf  $G$ -modules characterizes exactness. The vanishing theorem established here, somewhat surprisingly, disproves this conjecture.

The second application concerns fixed points for group actions. A classic result of M. M. Day [4] is a characterization of amenability via a fixed point property. Motivated by this fact we prove a fixed point theorem for actions of discrete groups on certain compact subsets of spaces of the  $\ell_\infty(G, X)$  type, equipped with a weak-type topology (Theorem 11). This topology, which we call the ultra-weak topology, is induced by  $\ell_\infty(G, X^*)$  viewed as maps into  $\ell_\infty(G)$  equipped with its weak-\* topology. This fixed point theorem can be viewed as a weak analogue of Day's theorem, which holds for all finitely generated groups.

Finally, in the last section we use the above results to define a notion of weak exactness for some Banach algebras (Definition 16). For  $C^*$ -algebras the notion of exactness is well-studied, see [2], and it would be interesting to try to extend such results to the setting of Banach algebras. One can compare this with the case of a  $C^*$ -algebra  $A$ , for which amenability of  $A$  as a Banach algebra is equivalent to nuclearity.

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## CONTENTS

1. Modules and topologies	3
2. Weak invariant expectations	4
3. Applications	5
References	10

## 1. MODULES AND TOPOLOGIES

**1.1. Actions.** Let  $G$  be a finitely generated group. A bounded Banach  $G$ -module is a Banach space  $X$  with a representation of  $G$  on  $X$ ,  $g \mapsto \pi_g$ , where each  $\pi_g$  is a bounded linear operator on  $X$ , satisfying  $\sup_{g \in G} \|\pi_g\| < \infty$ . Then the dual,  $X^*$ , is also a bounded Banach  $G$ -module with the representation  $\bar{\pi}_g = \pi_{g^{-1}}^*$ .

In general we denote the action of  $G$  on  $X$  by  $gx$ . Given a bounded Banach  $G$ -module  $X$ , we consider the action of  $G$  on  $\ell_\infty(G, X)$  defined by

$$(g * f)_h = g(f_{g^{-1}h}),$$

for  $g, h \in G$  and  $f \in \ell_\infty(G, X)$ . Then the induced action on  $\ell_\infty(G, \ell_\infty(G, X))$  will be denoted  $g \star f$  for  $f \in \ell_\infty(G, \ell_\infty(G, X))$  and  $g \in G$ .

**1.2. Pairings.** Let  $X$  be a Banach space. Denote by  $1_G$  the identity in  $\ell_\infty(G, X)$  and by  $\mathbb{1}_G$  the identity in  $\ell_\infty(G \times G)$ . Given a function  $\xi \in \ell_\infty(G \times G)$  we will view it as  $\xi \in \ell_\infty(G, \ell_\infty(G))$  by taking  $g \mapsto \xi(g, \cdot) = \xi_g \in \ell_\infty(G)$ . We then say that  $\xi$  is finitely supported if there exists a finite set  $F \subseteq G$  such that  $\xi_g = 0$  whenever  $g \in G \setminus F$ . That is, as a function on  $G \times G$ ,  $\xi$  is finitely supported in the first variable.

A finitely supported function  $\xi \in \ell_\infty(G, \ell_\infty(G))$  induces a bounded linear operator

$$\langle \xi, \cdot \rangle_{\ell_\infty(G)} : \ell_\infty(G, \ell_\infty(G, X)) \rightarrow \ell_\infty(G, X)$$

by the formula

$$\langle \xi, f \rangle_{\ell_\infty(G)} = \sum_{g \in G} \xi_g f_g.$$

Define the action of  $\ell_\infty(G)$  on  $\ell_\infty(G, X)$  by multiplication:

$$(a \bullet f)_g = a_g f_g,$$

and the action of  $\ell_\infty(G)$  on  $\ell_\infty(G, \ell_\infty(G, X))$

$$(af)_g = a \bullet \xi_g.$$

Then the operator  $\langle \xi, \cdot \rangle_{\ell_\infty(G)}$  is  $\ell_\infty(G)$ -linear, in the sense that

$$\langle \xi, af \rangle_{\ell_\infty(G)} = \langle a\xi, f \rangle_{\ell_\infty(G)} = a \langle \xi, f \rangle_{\ell_\infty(G)}.$$

For each  $g \in G$  we define the element  $\delta_g \in \ell_\infty(G, \ell_\infty(G))$  by setting

$$(\delta_g)_h = \begin{cases} 1_G & \text{if } g = h \\ 0 & \text{otherwise.} \end{cases}$$

Thus  $\mathbb{1}_G = \sum_{g \in G} \delta_g$ .

**1.3. The weak-\* operator topology on  $\mathcal{L}(X, M)$ .** We will denote weak-\* limits by  $w^* - \lim$ . Let  $X$  be a Banach space and  $M$  be a dual space. Consider the space  $\mathcal{L}(X, M)$  of bounded linear maps from  $X$  to  $M$ , with its natural operator norm, which we denote by  $\|\cdot\|_{\mathcal{L}}$ . Every element  $\xi \in X$  defines a map  $\hat{\xi} : \mathcal{L}(X, M) \rightarrow M$  by the formula

$$\hat{\xi}(T) = T(\xi)$$

for every  $T \in \mathcal{L}(X, M)$ . This defines a natural embedding

$$i : X \rightarrow \mathcal{L}(\mathcal{L}(X, M), M).$$

We denote the natural norm on  $\mathcal{L}(\mathcal{L}(X, M), M)$  by  $\|\cdot\|_{\mathcal{L}\mathcal{L}}$ . We have  $\|\hat{\xi}\|_{\mathcal{L}\mathcal{L}} = \|\xi\|_X$  for every  $\xi \in X$ . Let  $\widehat{B}_X \subseteq \mathcal{L}(\mathcal{L}(X, M), M)$  denote the image of the unit ball  $B_X$  of  $X$  under the inclusion  $i$ .

**Definition 2.** *The weak-\* operator topology on  $\mathcal{L}(X, M)$  is defined to be the weak-\* topology for which all operators in  $\widehat{B}_X$  are continuous when  $M$  is equipped with its weak-\* topology.*

Limits in the weak-\* operator topology on  $\mathcal{L}(X, M)$  will be denoted  $\mathcal{W}^* - \lim$ . The proof of the following lemma is analogous to the proof of the Banach-Alaoglu theorem.

**Lemma 3.** *The unit ball of  $\mathcal{L}(X, M)$  is compact in the weak-\* operator topology.*

We have the following description of the weak-\* operator topology on  $\mathcal{L}(X, M)$ .

**Proposition 4.** *Let  $X$  be a Banach space and let  $\{T_\beta\}$  be a net in  $\mathcal{L}(X, M)$ . The following conditions are equivalent:*

- (a)  $\mathcal{W}^* - \lim_\beta T_\beta = T$ ,
- (b)  $w^* - \lim_\beta T_\beta(x) = T(x)$  in  $M$  for every  $x \in X$ .

In the case  $Y = \ell_1(G)$  and  $Y^* = \ell_\infty(G)$  we can identify  $\mathcal{L}(X, \ell_\infty(G))$  with  $\ell_\infty(G, X^*)$ . The latter space is naturally the dual of  $\ell_1(G, X)$  and can be equipped with the corresponding weak-\* topology. The  $\mathcal{W}^*$ -topology and the weak-\* topology defined above agree on bounded subsets of  $\mathcal{L}(X, \ell_\infty(G))$ .

## 2. WEAK INVARIANT EXPECTATIONS

In [5] we proved a characterization of exactness in terms of invariant expectations; that is, operators whose properties are similar to properties of invariant means. We show that a weak version of such an operator always exists.

**Theorem 1.** *Let  $G$  be a finitely generated group and let  $X$  be a bounded Banach  $G$ -module. Then there exists a continuous linear map*

$$E : \ell_\infty(G, \ell_\infty(G, X^*)) \rightarrow \ell_\infty(G, X^*),$$

*called a weak invariant expectation on  $G$  with coefficients in  $X^*$ , such that*

- (1)  $E(g \star f) = g * (E(f))$  for every  $g \in G$  and  $f \in \ell_\infty(G, \ell_\infty(G, X^*))$ ,
- (2)  $E(af) = a \bullet E(f)$  for every  $a \in \ell_\infty(G)$  and  $f \in \ell_\infty(G, \ell_\infty(G, X^*))$ , and
- (3)  $E = \mathcal{W}^* - \lim_\beta \langle \xi_\beta, \cdot \rangle_{\ell_\infty(G)}$  in  $\mathcal{L}(\ell_\infty(G, \ell_\infty(G, X^*)), \ell_\infty(G, X^*))$ , where the  $\xi_\beta \in \ell_\infty(G, \ell_\infty(G))$  satisfy
  - (a) every  $\xi_\beta$  is finitely supported,
  - (b)  $\xi_\beta \geq 0$  as a function on  $G \times G$ , and
  - (c)  $\sum_{g \in G} (\xi_\beta)_g = 1_G$ .

*Proof.* Define  $E$  by the following formula:

$$(Ef)(g) = f(g, g),$$

where  $f \in \ell_\infty(G, \ell_\infty(G, X^*))$  is viewed as an element of  $\ell_\infty(G \times G, X^*)$ . It is easy to check that (1) and (2) are satisfied.

To prove the last property fix a finite generating set for  $G$ . Consider  $\Delta \in \ell_\infty(G \times G)$  defined by

$$\Delta(g, h) = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{otherwise.} \end{cases}$$

For a subset  $F \subseteq G$  denote by  $1_F$  the characteristic function of  $F$  and let  $B(n) \subset G$  denote the ball of radius  $n$  centered at the identity element. Let  $\xi_n : G \rightarrow \ell_\infty(G)$  be defined by

$$\xi_n = 1_{B(n)}\Delta + 1_{G \setminus B(n)}\delta_e.$$

The operators  $\langle \xi_n, \cdot \rangle_{\ell_\infty(G)}$  induced by the  $\xi_n$  are elements of the unit ball of the space  $\mathcal{L}(\ell_\infty(G \times G, X^*), \ell_\infty(G, X^*))$ .

For any finitely supported  $\eta \in \ell_1(G, X)$  we have

$$\langle \langle \xi_n, f \rangle_{\ell_\infty(G)}, \eta \rangle = \sum_{g \in \text{supp } \eta} (\langle \xi_n, f \rangle_{\ell_\infty(G)})_g (\eta_g)$$

Since the support of  $\eta$  is finite,  $\text{supp } \eta \subseteq B(n_0)$  for some  $n_0$ . Then for all  $n \geq n_0$  we have

$$\langle \langle \xi_n, f \rangle_{\ell_\infty(G)}, \eta \rangle = \langle Ef, \eta \rangle.$$

Therefore,

$$w^* - \lim_{n \rightarrow \infty} \langle \xi_n, f \rangle_{\ell_\infty(G)} = Ef,$$

in  $\ell_\infty(G, X^*)$ , which shows that  $E = \mathcal{W}^* - \lim \langle \xi_n, \cdot \rangle_{\ell_\infty(G)}$  and proves the claim.  $\square$

We remark that a weak invariant expectation with coefficients in  $X^* = \mathbb{R}$  equipped with a trivial  $G$ -action is a weak analogue of the invariant expectation considered in [5]. Indeed, in that case the domain of the weak invariant expectation is  $\ell_\infty(G, \ell_\infty(G)) \simeq \ell_\infty(G \times G)$ , which is a subspace of the space  $\mathcal{L}(\ell_u(G), \ell_\infty(G))$  of bounded linear maps from the uniform convolution algebra  $\ell_u(G)$  to  $\ell_\infty(G)$ .

### 3. APPLICATIONS

**I. Relative injectivity of Hopf  $G$ -modules.** Let  $X$  be a left Banach  $G$ -module.

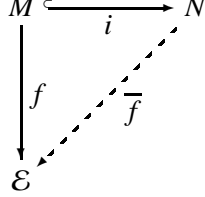
**Definition 5.** A subspace  $\mathcal{E} \subseteq \ell_\infty(G, X^*)$  is a Hopf  $G$ -module if it is both a  $G$ -submodule and an  $\ell_\infty(G)$ -submodule with respect to the actions  $*$  and  $\bullet$ , respectively.

Vanishing of bounded cohomology with coefficients in Hopf  $G$ -modules was studied in [5].

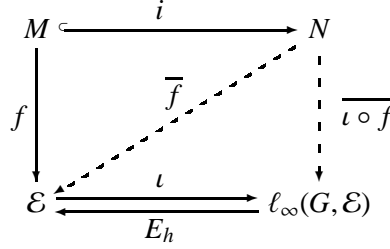
The notion of relative injectivity is a standard tool in the theory of Hochschild cohomology of Banach algebras and bounded cohomology of groups, see for example [6, 8, 10, 11], since it implies the vanishing of cohomology groups in all positive degrees. The definitions we use are from [8].

A continuous linear map  $f : M \rightarrow N$  between Banach spaces is admissible if there is a linear operator  $T : N \rightarrow M$  such that  $\|T\| \leq 1$  and  $fTf = f$ . We assume that all  $G$ -module maps between bounded Banach  $G$ -modules are continuous.

**Definition 6.** A bounded Banach  $G$ -module  $\mathcal{E}$  is relatively injective if for every injective admissible  $G$ -morphism  $i : M \rightarrow N$  and any  $G$ -morphism  $f : M \rightarrow \mathcal{E}$  there is a  $G$ -morphism  $\bar{f} : N \rightarrow \mathcal{E}$  such that  $\bar{f} \circ i = f$  and  $\|\bar{f}\| \leq \|f\|$ .



For a Banach  $G$ -module  $\mathcal{E}$  the module  $\ell_\infty(G, \mathcal{E})$  is relatively injective [8]. If the injection  $\iota : \mathcal{E} \rightarrow \ell_\infty(G, \mathcal{E})$ ,  $\iota(x) = x1_G$ , admits a right inverse  $E_h$  of norm 1 which commutes with the action of  $G$  then the module  $\mathcal{E}$  is also relatively injective. Indeed, given the diagram

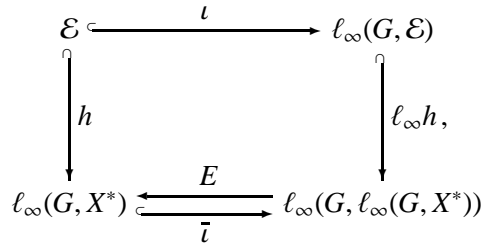


one verifies that  $E_h \circ (\overline{\iota \circ f}) \circ i = f$  and that  $\|E_h \circ \overline{\iota \circ f}\| = \|f\|$ .

We now use the weak invariant expectation to show that Hopf  $G$ -modules satisfy the conditions of Definition 6.

**Theorem 7.** Every weak- $*$  closed Hopf  $G$ -module is a relatively injective  $G$ -module.

*Proof.* Consider the following diagram:



where  $h$  is the natural Hopf inclusion of  $\mathcal{E}$  into  $\ell_\infty(G, X^*)$  for some  $G$ -module  $X$ ,  $\ell_\infty h$  is induced by applying  $h$  coordinate-wise and  $E$  is a weak invariant expectation. Define  $E_h : \ell_\infty(G, \mathcal{E}) \rightarrow \ell_\infty(G, X^*)$  by

$$E_h = E \circ \ell_\infty h.$$

In that case, the explicit formula for  $E$  yields

$$(1) \quad (E_h \eta)_g = \left( h(\eta_g) \right)_g,$$

for every  $\eta \in \ell_\infty(G, \mathcal{E})$ .

By the properties of  $E$ , for every  $\eta \in \ell_\infty(G, \mathcal{E})$  we have

$$E_h(\eta) = w^* - \lim_n \sum_{g \in G} (\xi_n) \eta_g,$$

where the  $\xi_n$  are as in Theorem 1. Since  $\eta_g \in \mathcal{E}$ ,  $\xi_n$  is finitely supported and  $\mathcal{E}$  is closed under the action of  $\ell_\infty(G)$ , we have that  $\sum_{g \in G} (\xi_n) \eta_g$  is an element of  $\mathcal{E}$  for every  $\beta$ . Also,  $\mathcal{E}$  is weak-\* closed and thus the limit belongs to  $\mathcal{E}$ .

Additionally, for  $x \in \mathcal{E}$  it follows from (1) that

$$E_h(u(x)) = E(x1_G) = x,$$

for every  $x \in \mathcal{E}$ . The fact that  $E_h$  is  $G$ -equivariant follows from the properties of  $E$  and the fact that  $E_h$  is a restriction of  $E$  to a  $G$ -invariant subspace. Finally, it is also easy to verify that  $\|E_h\| = \|E\| = 1$ .  $\square$

Theorem 7 allows one to deduce a vanishing theorem for bounded cohomology with coefficients in Hopf  $G$ -modules.

**Theorem 8.** *Let  $G$  be a finitely generated group. Then the bounded cohomology  $H_b^n(G, \mathcal{E})$  vanishes for every  $n \geq 1$  and every weak-\* closed Hopf  $G$ -module  $\mathcal{E}$ .*

Theorem 8 follows from [8, Proposition 7.4.1] and Theorem 7.

**II. A fixed point theorem for actions on  $\ell_\infty(G, X)$ .** The existence of a weak invariant expectation allows one to prove a fixed point theorem for a group acting on spaces of the type  $\ell_\infty(G, X)$ , where  $X$  is a normed space. The fixed point theorem we prove can be viewed as a weak analogue of Day's classical fixed point theorem for amenable groups [4].

**Definition 9.** *A subset  $K \subseteq \ell_\infty(G, X)$  is called  $\ell_\infty(G)$ -convex if given any finite collection of positive elements  $a_1, \dots, a_n \in \ell_\infty(G)$  such that  $\sum a_i = 1_G$ , we have  $\sum a_i x_i \in K$  for any  $x_1, \dots, x_n \in K$ .*

We equip  $\ell_\infty(G, X)$  with a topology as follows. Every  $\varphi \in \ell_\infty(G, X^*)$  induces a bounded linear operator  $T_\varphi : \ell_\infty(G, X) \rightarrow \ell_\infty(G)$  by the formula

$$T_\varphi f(g) = \langle \varphi_g, f_g \rangle.$$

In particular, the inclusion  $i : X^* \rightarrow \ell_\infty(G, X^*)$  as the constant functions allows one to interpret each element of  $X^*$  as such an operator.

**Definition 10.** *Let  $V \subseteq X^*$  be a weak-\* dense subspace. The ultra-weak topology induced by  $V$  on  $\ell_\infty(G, X)$  is the weakest topology with respect to which every operator  $T_\varphi$  induced by  $\varphi \in V$  is continuous, when  $\ell_\infty(G)$  is equipped with its natural weak-\* topology.*

We will usually omit the reference to  $V$ . One important property of the operators induced by elements of  $V$  is that they separate the points of  $\ell_\infty(G, X)$ . This property is crucial in our argument. Note also that if  $X = Y^*$  is itself a dual space, then we can take  $V = X \subset X^{**}$ . In that case the ultra-weak topology on  $\ell_\infty(G, Y^*)$  is precisely the  $\mathcal{W}^*$ -topology on  $\ell_\infty(G, Y^*)$ , in the sense of the previous sections.

An action of a group  $G$  on a subset  $K \subseteq \ell_\infty(G, X)$  is said to be  $G$ -affine if

$$g(ax + by) = (g * a)gx + (g * b)gy$$

for  $g \in G$ ,  $x, y \in K$  and  $a, b \in \ell_\infty(G)$  such that  $a, b \geq 0$  and  $a + b = 1_X$ . Note that such an action is not, in general, inherited from an action on  $\ell_\infty(G, X)$ .

**Theorem 11.** *Let  $G$  be a finitely generated group,  $X$  be a Banach space and  $V \subseteq X^*$  be a weak-\* dense subspace. Then every  $G$ -affine action of  $G$  on a bounded,  $\ell_\infty(G)$ -convex, ultra-weakly compact subset  $K \subseteq \ell_\infty(G, X)$  has a fixed point.*

*Proof.* We divide the proof into a few lemmas, with the assumptions for each of them being the same. Fix  $\kappa_0 \in K$ . Let  $\mathcal{A}(K, \ell_\infty(G))$  denote the set of all weak-\* continuous maps  $T : K \rightarrow \ell_\infty(G)$  which are  $\ell_\infty(G)$ -convex; that is,

$$T(ax + by) = aT(x) + bT(y)$$

for  $x, y \in K$  and  $a, b \in \ell_\infty(G)$ ,  $a \geq 0$ ,  $b \geq 0$  and  $a + b = 1_G$ . Observe that  $\ell_\infty(G, X^*) \subseteq \mathcal{A}(K, \ell_\infty(G))$  when restricted to  $K$ .

The space  $\mathcal{A}(K, \ell_\infty(G))$  plays, roughly speaking, the role of a “dual space with coefficients in  $\ell_\infty(G)$ ”. Given  $T \in \mathcal{A}(K, \ell_\infty(G))$  and  $g \in G$  define

$$g \cdot T(x) = g * T(g^{-1}x),$$

for every  $x \in K$ .

**Lemma 12.** *The operation  $\cdot$  defines an action of  $G$  on  $\mathcal{A}(K, \ell_\infty(G))$ .*

*Proof.* We only need to show that  $g \cdot T$  is  $\ell_\infty(G)$ -convex. For  $a, b \in \ell_\infty(G)$  such that  $a \geq 0$ ,  $b \geq 0$ ,  $a + b = 1_G$  and  $x, y \in K$ , we have

$$\begin{aligned} (g \cdot T)(ax + by) &= g * (T(g^{-1}(ax + by))) \\ &= g * ((g^{-1} * a)T(g^{-1}x) + (g^{-1} * b)T(g^{-1}y)) \\ &= a(g * T(g^{-1}x)) + b(g * T(g^{-1}y)) \\ &= a(g \cdot T)(x) + b(g \cdot T)(y). \end{aligned}$$

□

For every  $T \in \mathcal{A}(K, \ell_\infty(G))$  define  $f_{[T]} : G \rightarrow \ell_\infty(G)$  by the formula

$$f_{[T]}(g) = T(g\kappa_0).$$

We have  $f_{[T]} \in \ell_\infty(G, \ell_\infty(G))$ .

**Lemma 13.** *There exists a point  $x \in K$  such that  $E(f_{[T]}) = T(x)$  for every  $T \in \mathcal{A}(K, \ell_\infty(G))$ .*



*Proof.* Since  $E = \mathcal{W}^* - \lim_{\beta} \langle \xi_{\beta}, \cdot \rangle_{\ell_{\infty}(G)}$  we have

$$\begin{aligned} \langle \xi_{\beta}, f_{[T]} \rangle_{\ell_{\infty}(G)} &= \sum_{g \in G} (\xi_{\beta})_g (f_{[T]})_g \\ &= \sum_{g \in G} (\xi_{\beta})_g T(g\kappa_0) \\ &= \sum_{g \in G} T((\xi_{\beta})_g g\kappa_0) \\ &= T\left(\sum_{g \in G} (\xi_{\beta})_g g\kappa_0\right) \\ &= T(x_{\beta}), \end{aligned}$$

where we used the fact that  $T$  is  $\ell_{\infty}(G)$ -linear and that the  $\xi_{\beta}$  are finitely supported. By the ultra-weak compactness of  $K$  there exists a convergent subnet of the  $x_{\beta}$ , which we denote again by  $x_{\beta}$ , and we define  $x_0 = \lim_{\beta} x_{\beta}$ . Then for  $T \in \mathcal{A}(K, \ell_{\infty}(G))$  we have

$$T(x_0) = w^* - \lim_{\beta} T(x_{\beta}) = w^* - \lim_{\beta} \langle f_{[T]}, \xi_{\beta} \rangle_{\ell_{\infty}(G)} = E(f_{[T]}),$$

by the ultra-weak continuity of  $T$ .  $\square$

**Lemma 14.** For  $g \in G$  we have  $f_{[g \cdot T]} = g \star f_{[T]}$ .

*Proof.* For every  $h \in G$  we have

$$\begin{aligned} (f_{[g \cdot T]})_h &= (g \cdot T)(h\kappa_0) \\ &= g * (T(g^{-1}h\kappa_0)) \\ &= g * ((f_{[T]})_{g^{-1}h}) \\ &= (g \star f_{[T]})_h. \end{aligned}$$

$\square$

We now verify that  $x_0$  is a fixed point. For an operator  $T \in V \subseteq \mathcal{A}(K, \ell_{\infty}(G))$  we obtain

$$\begin{aligned} T(gx_0) &= g * (g^{-1} \cdot T)(x_0) \\ &= g * E(f_{[g^{-1} \cdot T]}) \\ &= g * E(g^{-1} \star f_{[T]}) \\ &= E(f_{[T]}) \\ &= T(x_0). \end{aligned}$$

Since elements of  $V$  separate points of  $K$ , it follows that  $gx_0 = x_0$  and  $x_0$  is a fixed point, which completes the proof of Theorem 11.  $\square$

We expect that the above fixed point theorem can be generalized to semigroups.

**III. Weakly exact Banach algebras.** The above results on bounded cohomology of groups suggest one might define a notion of weak exactness for certain Banach algebras. Such algebras have to be co-algebras in an appropriate sense, so that their duals are Banach algebras in a natural way as well. This requirement is a consequence of the fact that we have used the structure of  $\ell_1(G)$  as a Hopf algebra, not only as a Banach algebra. We will consider only preduals of von Neumann algebras but it is clear that the definition can be extended to other cases.

Let  $M$  be a Hopf-von Neumann algebra and let  $A = M_*$  denote a predual Banach algebra. Let  $X$  be a right  $A$ -module and consider the space  $\mathcal{L}(X, M)$ . The algebra  $M$  is an  $A$ -bimodule in a natural way, as it is the dual of the  $A$ -bimodule  $A$ . Thus  $\mathcal{L}(X, M)$  is an  $A$ -bimodule with the following actions:

$$\begin{aligned}(a \cdot T)(x) &= T(xa), \\ (T \cdot a)(x) &= T(x)a,\end{aligned}$$

for  $a \in A$ ,  $T \in \mathcal{L}(X, M)$  and  $x \in X$ . Since  $M$  is an algebra, there is the additional structure of an  $M$ -module on  $\mathcal{L}(X, M)$  given by

$$(bT)(x) = b(T(x)),$$

for  $b \in M$ ,  $T \in \mathcal{L}(X, M)$  and  $x \in X$ .

**Definition 15.** *Let  $M$  be a Hopf-von Neumann algebra and  $A$  its predual Banach algebra. A submodule of  $\mathcal{L}(X, M)$ , which is both an  $A$ -bimodule and an  $M$ -module with respect to the structures described above, is called a Hopf  $A$ -bimodule.*

Recall that given a Banach algebra  $A$  and an  $A$ -bimodule  $\mathcal{E}$  one can define the Hochschild cohomology groups  $\mathcal{H}^*(A, \mathcal{E})$  of  $A$  with coefficients in  $\mathcal{E}$ . In particular, the first cohomology group  $\mathcal{H}^1(A, \mathcal{E})$  is defined as the quotient of the space of all  $A$ -derivations from  $A$  into  $X$  modulo the inner derivations, see for example [3, 10].

**Definition 16.** *Let  $M$  be a Hopf-von Neumann algebra and let  $A$  be a predual Banach algebra of  $M$ . We define  $A$  to be weakly exact if*

$$\mathcal{H}^1(A, \mathcal{E}) = 0$$

*for every  $M$ -submodule  $\mathcal{E} \subseteq \mathcal{L}(X, M)$ , which is closed in the weak-\* operator topology, where  $X$  is any left  $A$ -module.*

It is natural to ask if dimension shifting preserves the class of Hopf modules over  $A$  and, more importantly, do algebras behave similarly to finitely generated groups:

**Question 17.** *Is every Banach algebra  $A$  as above weakly exact?*

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